

MAS212 Linear Algebra I 2007-2008

Coursework 9 — Solutions

Exercise* 1. Let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Note that

$$\|\mathbf{x} + \mathbf{y}\|^2 = (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{x} \cdot (\mathbf{x} + \mathbf{y}) + \mathbf{y} \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{x} + \mathbf{y} \cdot \mathbf{y} = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\mathbf{x} \cdot \mathbf{y}.$$

Thus $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$, that is, $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ if and only if $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

Exercise* 2.

- (a) Let $\mathbf{x} = (x_1, x_2, x_3)^T$. By Exercise 3, the vector $\mathbf{x}$ is in $S^\perp$ if and only if $\mathbf{x}$ is orthogonal to $\mathbf{y}$, that is, $\mathbf{y}^T \mathbf{x} = 0$. Thus $S^\perp$ is the nullspace of the 1×3 matrix

$$\mathbf{y}^T = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix},$$

which is in row echelon form, with leading variable x_1 and free variables x_2 and x_3 . Thus, setting $x_2 = \alpha$ and $x_3 = \beta$ we find $x_1 = -x_2 + x_3 = -\alpha + \beta$. Hence, any $\mathbf{x} \in S^\perp$ is of the form

$$\begin{pmatrix} -\alpha + \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

so $\{(-1, 1, 0)^T, (1, 0, 1)^T\}$ is a basis for $S^\perp$.

- (b) In geometric terms S can be described as a line through the origin in the direction of $\mathbf{y}$, while $S^\perp$ can be described as a plane through the origin perpendicular to $\mathbf{y}$.

Exercise* 3. If $\mathbf{x}$ belongs to $S^\perp$ then $\mathbf{x}$ is perpendicular to any vector in S , so in particular to the vectors that span S .

Conversely, suppose that $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$ for $j = 1, \dots, r$, that is,

$$\mathbf{x} \cdot \mathbf{v}_j = 0 \quad \text{for all } j = 1, \dots, r.$$

We need to show that $\mathbf{x}$ is orthogonal to every vector in S . Now, if $\mathbf{v}$ is an arbitrary vector in S , then there are scalars $\alpha_1, \dots, \alpha_r$ such that

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_r \mathbf{v}_r.$$

But then

$$\mathbf{x} \cdot \mathbf{v} = \mathbf{x} \cdot (\alpha_1 \mathbf{v}_1 + \dots + \alpha_r \mathbf{v}_r) = \alpha_1 (\mathbf{x} \cdot \mathbf{v}_1) + \dots + \alpha_r (\mathbf{x} \cdot \mathbf{v}_r) = 0 + \dots + 0 = 0,$$

so $\mathbf{x}$ is indeed orthogonal to $\mathbf{v}$.

Exercise 4.

- (a) First observe that $\mathbf{0} \in S^\perp$ since $\mathbf{0}$ is orthogonal to every vector in $\mathbb{R}^n$, so, in particular, to all vectors in S . Thus $S^\perp$ is not empty. Let now $\mathbf{x}_1, \mathbf{x}_2 \in S^\perp$ and let $\alpha_1, \alpha_2 \in \mathbb{R}$. Then

$$\mathbf{x}_1 \cdot \mathbf{y} = 0 \quad \text{for every } \mathbf{y} \in S,$$

$$\mathbf{x}_2 \cdot \mathbf{y} = 0 \quad \text{for every } \mathbf{y} \in S,$$

so, for any $\mathbf{y} \in S$,

$$(\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2) \cdot \mathbf{y} = \alpha_1 (\mathbf{x}_1 \cdot \mathbf{y}) + \alpha_2 (\mathbf{x}_2 \cdot \mathbf{y}) = 0 + 0 = 0,$$

and hence $\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 \in S^\perp$. Thus $S^\perp$ is a subspace of $\mathbb{R}^n$ as claimed.

(b) If $\mathbf{x} \in S$, then $\mathbf{x}$ is orthogonal to each $\mathbf{y} \in S^\perp$, so $\mathbf{x} \in (S^\perp)^\perp$. Thus $S \subset (S^\perp)^\perp$.

Conversely, suppose that $\mathbf{x} \in (S^\perp)^\perp$. By the Orthogonal Decomposition Theorem we can write $\mathbf{x}$ as a sum $\hat{\mathbf{x}} + \mathbf{z}$, where $\hat{\mathbf{x}} \in S$ and $\mathbf{z} \in S^\perp$. Since $\mathbf{z} \in S^\perp$, it is orthogonal to both $\hat{\mathbf{x}}$ and $\mathbf{x}$, so

$$0 = \mathbf{z} \cdot \mathbf{x} = \mathbf{z} \cdot \hat{\mathbf{x}} + \mathbf{z} \cdot \mathbf{z} = \mathbf{z} \cdot \mathbf{z}.$$

Thus $\mathbf{z} = \mathbf{0}$, which implies that $\mathbf{x} = \hat{\mathbf{x}} \in S$. It follows that $(S^\perp)^\perp \subset S$.

Exercise* 5. Let $\mathbf{x}$ be in the column space of A and suppose that $A^T \mathbf{x} = \mathbf{0}$. Thus $\mathbf{x}$ belongs to both the column space of A and the nullspace of A^T . But since, by the Fundamental Subspace Theorem, these two spaces are orthogonal complements of each other, the vector $\mathbf{x}$ must be orthogonal to itself. Thus $\mathbf{x} \cdot \mathbf{x} = 0$, which implies that $\mathbf{x} = \mathbf{0}$.

Exercise* 6. Since

$$\mathbf{y} \cdot \mathbf{u} = 3 \cdot 2 + 4 \cdot 1 = 10$$

$$\mathbf{u} \cdot \mathbf{u} = 2 \cdot 2 + 1 \cdot 1 = 5$$

the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto $\mathbf{u}$ is given by

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{10}{5} \mathbf{u} = 2\mathbf{u} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

Thus

$$\hat{\mathbf{y}} - \mathbf{y} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

But

$$(\hat{\mathbf{y}} - \mathbf{y}) \cdot \mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = (-1) \cdot 2 + 2 \cdot 1 = 0$$

so $(\hat{\mathbf{y}} - \mathbf{y})$ and $\mathbf{u}$ are indeed orthogonal (as they should!). Now, since $\hat{\mathbf{y}} - \mathbf{y}$ is perpendicular to $\mathbf{u}$ and $\|\hat{\mathbf{y}} - \mathbf{y}\|$ is the distance from $\hat{\mathbf{y}}$ to $\mathbf{y}$, the norm of $\hat{\mathbf{y}} - \mathbf{y}$ is the shortest distance from $\mathbf{y}$ to the line in the direction $\mathbf{u}$.

Exercise* 7.

(a) Observe that by a theorem proved in the lectures $U^T U = I$, since U has orthonormal columns. Thus

$$(U\mathbf{x}) \cdot (U\mathbf{y}) = (U\mathbf{x})^T U\mathbf{y} = \mathbf{x}^T U^T U\mathbf{y} = \mathbf{x}^T I\mathbf{y} = \mathbf{x}^T \mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$

(b) Notice that by (a) we have

$$\|U\mathbf{x}\|^2 = (U\mathbf{x}) \cdot (U\mathbf{x}) = \mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2,$$

so the statement follows by taking square roots.

(c) Again by (a), we see that $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$, so $U\mathbf{x}$ is orthogonal to $U\mathbf{y}$ if and only if $\mathbf{x}$ is orthogonal to $\mathbf{y}$.

Exercise* 8.

(a) Since

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = 1 \cdot 2 + (-1) \cdot 0 + 2 \cdot (-1) = 0$$

$$\mathbf{u}_1 \cdot \mathbf{u}_3 = 1 \cdot 1 + (-1) \cdot 5 + 2 \cdot 2 = 0$$

$$\mathbf{u}_2 \cdot \mathbf{u}_3 = 2 \cdot 1 + 0 \cdot 5 + (-1) \cdot 2 = 0$$

$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set. But, by a theorem proved in the lectures, vectors of an orthogonal set are linearly independent, so $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis for $\mathbb{R}^3$, since $\dim \mathbb{R}^3 = 3$.

(b) First we calculate the following dot products:

$$\mathbf{y} \cdot \mathbf{u}_1 = 3 \cdot 1 + 7 \cdot (-1) + 11 \cdot 2 = 18$$

$$\mathbf{u}_1 \cdot \mathbf{u}_1 = 1^2 + (-1)^2 + 2^2 = 6$$

$$\mathbf{y} \cdot \mathbf{u}_2 = 3 \cdot 2 + 7 \cdot 0 + 11 \cdot (-1) = -5$$

$$\mathbf{u}_2 \cdot \mathbf{u}_2 = 2^2 + 0^2 + (-1)^2 = 5$$

$$\mathbf{y} \cdot \mathbf{u}_3 = 3 \cdot 1 + 7 \cdot 5 + 11 \cdot 2 = 60$$

$$\mathbf{u}_3 \cdot \mathbf{u}_3 = 1^2 + 5^2 + 2^2 = 30$$

Thus

$$\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2 + \frac{\mathbf{y} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3} \mathbf{u}_3 = \frac{18}{6} \mathbf{u}_1 + \frac{-5}{5} \mathbf{u}_2 + \frac{60}{30} \mathbf{u}_3 = 3\mathbf{u}_1 - \mathbf{u}_2 + 2\mathbf{u}_3.$$

(c) Since normalising a set of vectors does not change their span, we obtain an orthonormal basis for $\mathbb{R}^3$ with the stated properties by defining

$$\mathbf{v}_1 = \frac{1}{\|\mathbf{u}_1\|} \mathbf{u}_1 = \begin{pmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}, \quad \mathbf{v}_2 = \frac{1}{\|\mathbf{u}_2\|} \mathbf{u}_2 = \begin{pmatrix} 2/\sqrt{5} \\ 0 \\ -1/\sqrt{5} \end{pmatrix}, \quad \mathbf{v}_3 = \frac{1}{\|\mathbf{u}_3\|} \mathbf{u}_3 = \begin{pmatrix} 1/\sqrt{30} \\ 5/\sqrt{30} \\ 2/\sqrt{30} \end{pmatrix}.$$

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